Georg Fantner

Strain and stress in higher dimensions

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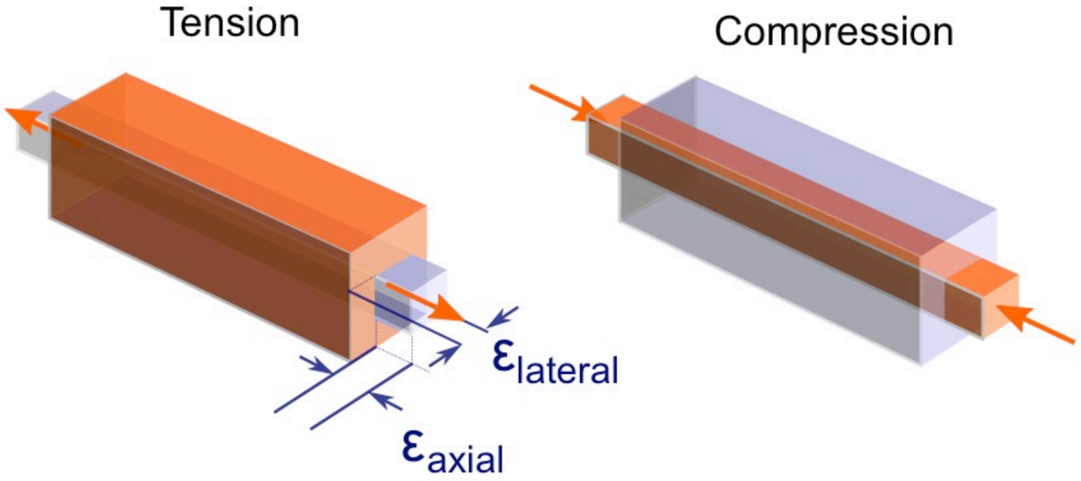
Poisson Effect

DEMO.

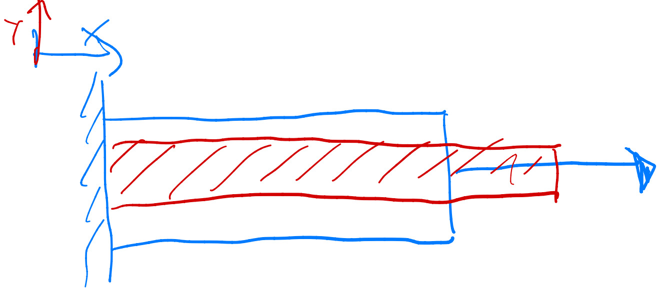
3D response to 1D strain

The Poisson effect

- In chapter 2 we've discussed how loads introduce stress and strain in the the direction that the load acts (1D problems)
- In reality, a strain in one direction (*axial strain*) also induces strains in the two perpendicular directions (*lateral strain*)



3D response to 1D strain
The Poisson effect



$$\boxed{\varepsilon_{lateral} = -\nu \cdot \varepsilon_{axial}}$$

$$\nu := \left| \frac{\varepsilon_{lateral}}{\varepsilon_{axial}} \right| = \text{Poisson ratio}$$

$$\boxed{G = \frac{E}{2(1+\nu)}}$$

The lateral strain can be described using the *Poisson equation.*

ν is a materials property and is on the order of 0.25-0.35 for most materials (0.1-0.5)

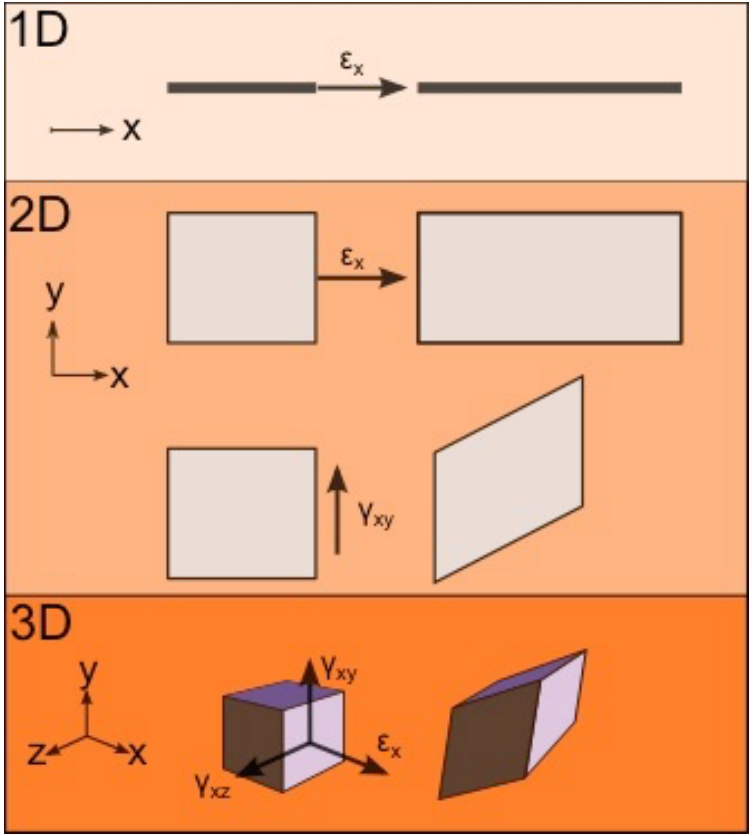
IMPORTANT: The Poisson effect does NOT cause any additional stress in the material, unless the transverse displacement is prevented

For Hookean materials, the Poisson ratio relates the *elastic modulus E* and the *shear modulus G.*

"Where did you get yor chiropractic license?"

The Strain Tensor

The strain tensor



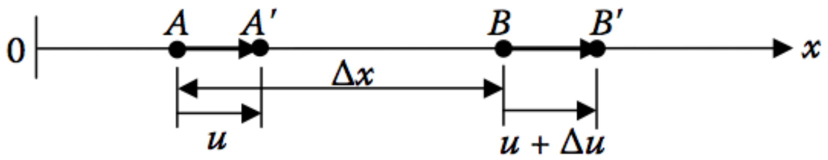
In real world problems, we need to keep track of normal strains ε and shear strains γ in multiple directions

On each face of a (virtual) cube, we can have one normal strain and two shear strains

For 3 dimensions (3 faces) this results in 3 normal strains and 6 shear strains

The 9 strains can be conveniently combined into the strain tensor

REMEMBER: Strain is a local property and can change dramatically in the material. That's why we need a definition of the 3D strain based on an infinitesimal element.



$$\boxed{\varepsilon = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{du}{dx}}$$

Review: The strain tensor

Infinitesimal definition of strain

- Like in the 1D case, we derive the definition of strain from the stretching of a segment AB with the original length of Δx that is part of a larger line.
- *u* is the “rigid body motion” and *Δu* is the stretching of the element *Δx*

The strain tensor

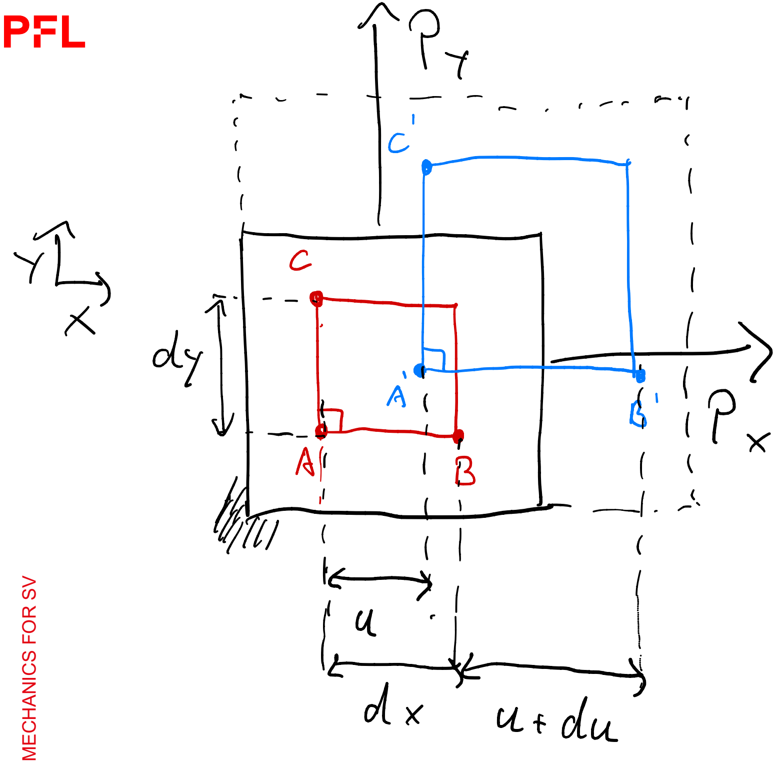
Definition of *normal* strain in 3 dimensions

- The three normal components of the strain are then:

$$\begin{aligned} \varepsilon_x &= \varepsilon_{xx} = \frac{\partial u}{\partial x} \\ \varepsilon_y &= \varepsilon_{yy} = \frac{\partial v}{\partial y} \\ \varepsilon_z &= \varepsilon_{zz} = \frac{\partial w}{\partial z} \end{aligned}$$

- The 3 normal changes show the change in shape of the parallelepiped with initial volume *dxdydz.*
- The normalized volume change is the sum of the normalized strains

$$\frac{\Delta V}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z$$



$P_x \perp P_y \Rightarrow$ ONLY NORMAL FORCES

u := DISPLACEMENT IN x DIRECTION

v := " " y "

$$du = \frac{\partial u}{\partial x} dx$$

$$\varepsilon_x = \frac{\Delta L_x}{L_x} = \frac{\Delta L_x}{dx}$$

$$L_x' = dx + \cancel{u} + du - \cancel{u} = dx + \frac{\partial u}{\partial x} dx$$

$$\varepsilon_x = \frac{\Delta L_x}{dx} = \frac{\cancel{dx} + \frac{\partial u}{\partial x}\cancel{dx} - \cancel{dx}}{\cancel{dx}} = \frac{\partial u}{\partial x}$$

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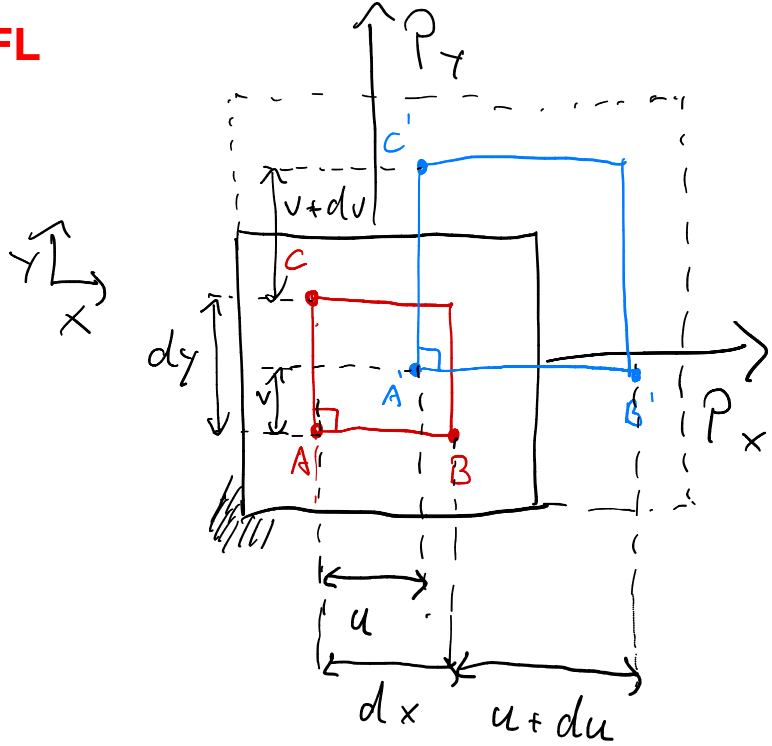


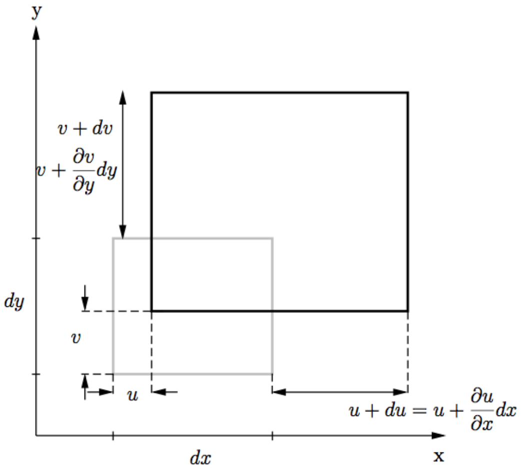
IN Y:

$$dv = \frac{\partial v}{\partial y} dy$$

$$L_y' = dy + \cancel{v} + dv - \cancel{v} = dy + \frac{\partial v}{\partial y} dy$$

$$\varepsilon_y = \frac{\Delta L_y}{L_y} = \frac{\cancel{dy} + \frac{\partial v}{\partial y} \cancel{dy} - \cancel{dy}}{\cancel{dy}}$$

$$\boxed{\varepsilon_y = \frac{\partial v}{\partial y}}$$



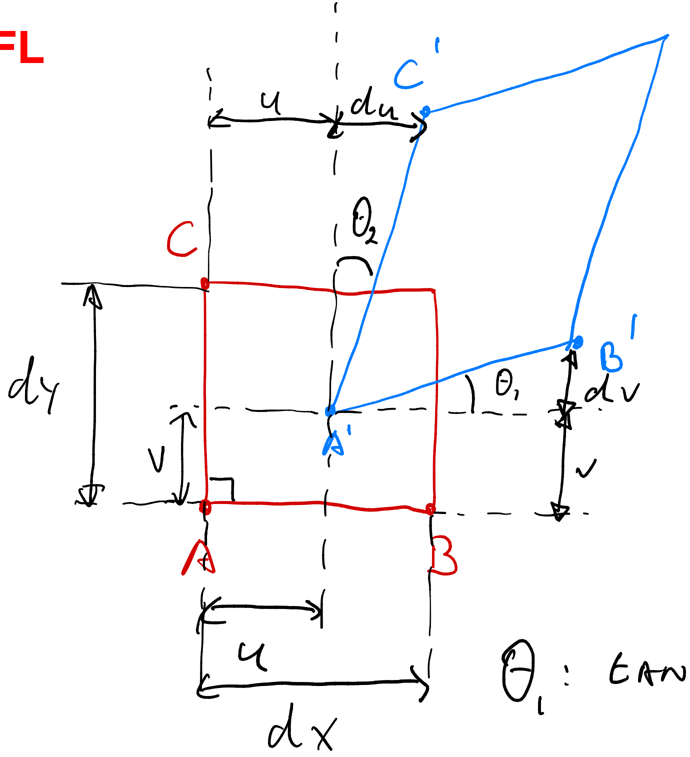
$$\varepsilon_x = \varepsilon_{xx} = \frac{\partial u}{\partial x}$$
$$\varepsilon_y = \varepsilon_{yy} = \frac{\partial v}{\partial y}$$
$$\varepsilon_z = \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

$$\frac{\Delta V}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

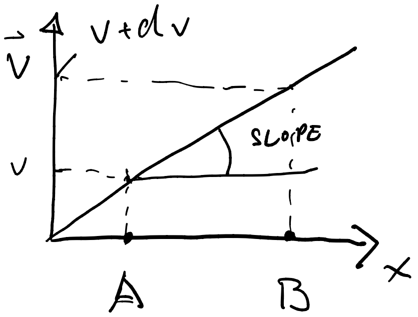
Deriving the strain tensor

Definition of normal strain in 3 dimensions

- Analog to the 1D case we can define the 3 normal components of the strain.
- The 3 normal changes show the change in shape of the parallelepiped with initial volume *dxdydz.*
- The normalized volume change is the sum of the normalized strains



LET US PLOT V AS A FUNCTION OF X



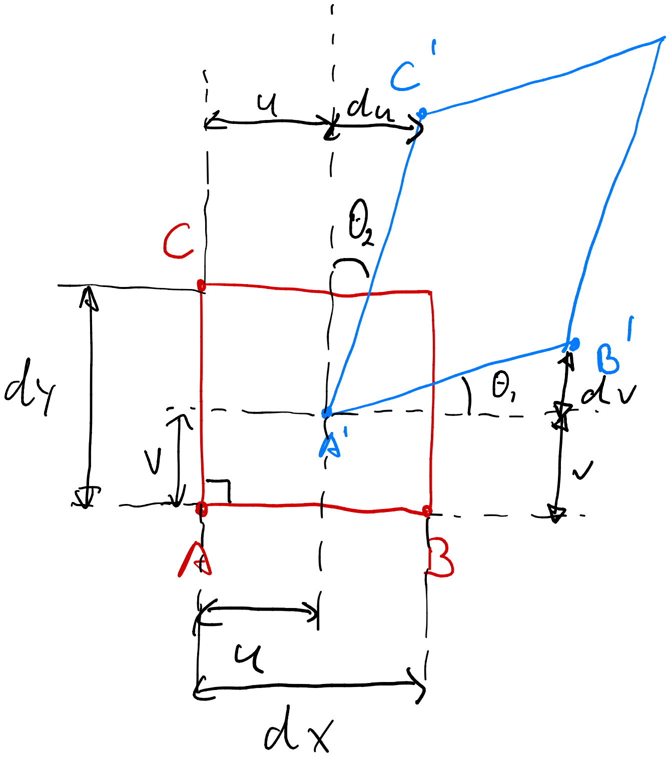
$$\text{SLOPE} := \frac{\partial v}{\partial x} = \frac{\vec{v}(B) - \vec{v}(A)}{B - A}$$

$$= \frac{dv}{dx}$$

$$dv = \frac{\partial v}{\partial x} \cdot dx$$

$$\theta_1 : \tan \theta_1 = \frac{dv}{L'_x} = \frac{dv}{dx + du} = \frac{\frac{\partial v}{\partial x} \cdot \cancel{dx}}{\cancel{dx} + \frac{\partial u}{\partial x} \cancel{dx}}$$

$$\tan \theta_1 = \frac{\frac{\partial v}{\partial x}}{1 + \underbrace{\cancel{\frac{\partial u}{\partial x}}}_{\ll 1}} = \frac{\partial v}{\partial x}$$



$$\theta_2 : \tan \theta_2 = \frac{du}{L'_y} = \frac{du}{dy + dv}$$

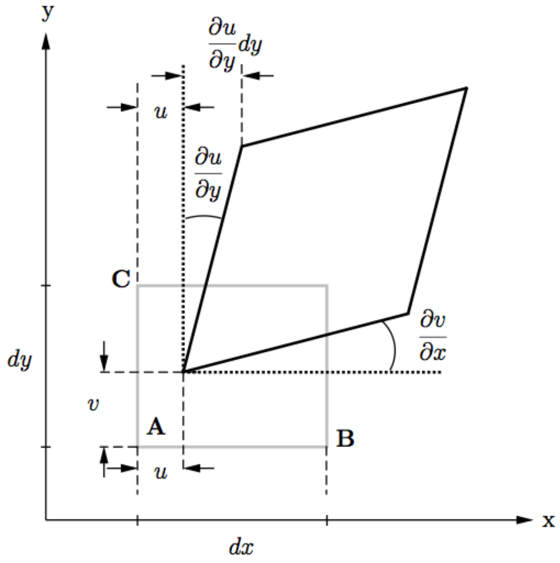
$$= \frac{\frac{\partial u}{\partial y} \cancel{dy}}{\cancel{dy} + \underbrace{\frac{\partial v}{\partial y}}_{\ll 1} \cancel{dy}} = \frac{\partial u}{\partial y}$$

SHEAR STRAIN γ_{xy} :

$$\boxed{\gamma_{xy} = \theta_1 + \theta_2 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}}$$

The strain tensor

Definition of shear strain in 3 dimensions



- The slopes of initially horizontal line is $\frac{\partial v}{\partial x}$
- The slopes of initially vertical line is $\frac{\partial u}{\partial y}$
- Therefore the change in the previously right angle, which is the shear strain is:

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

- From symmetry and extrapolation into the third dimension we learn that:

$$\boxed{\begin{aligned} \gamma_{xy} = \gamma_{yx} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \\ \gamma_{xz} = \gamma_{zx} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \\ \gamma_{yz} = \gamma_{zy} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \end{aligned}}$$

ε_{xx}

ε_{yy}

ε_{zz}

The strain tensor

Nomenclature

$$\overleftrightarrow{\varepsilon} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} = \begin{pmatrix} \varepsilon_{xx} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_{yy} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_{zz} \end{pmatrix} =$$

- The shear components of the strain tensor are DEFINED to be ½ of the engineering strain.
- THIS IS AN ENDLESS SOURCE OF CONFUSION!

$$\overleftrightarrow{\varepsilon} = \begin{pmatrix} \varepsilon_{xx} & \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \varepsilon_{yy} & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & \varepsilon_{zz} \end{pmatrix}$$

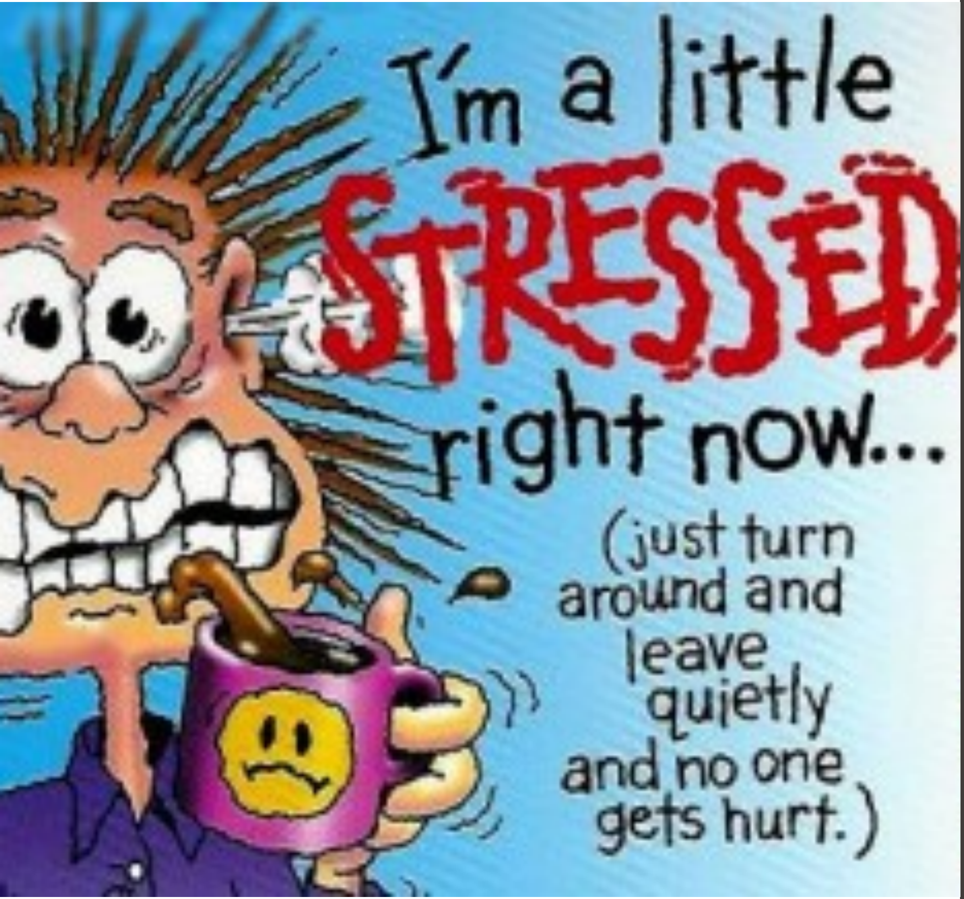
The strain tensor

Nomenclature

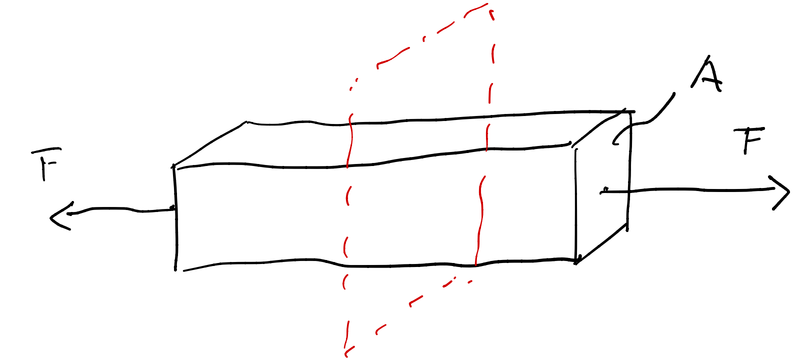
- We can write the 9 components of the strain as a second order tensor (for each strain we have a direction, a magnitude and relative to which plane our stress is quantified).
- Example: ε_{xx} represents the magnitude of deformation in the x direction, relative to a reference length in the x direction.
- To make the strain tensor behave mathematically like a proper tensor, the shear components of the strain tensor are defined as HALF the engineering strain.

$$\varepsilon_{kl} = \frac{1}{2}(u_{l,k} + u_{k,l}) = \frac{1}{2}\left(\frac{\partial u_l}{\partial x_k} + \frac{\partial u_k}{\partial x_l}\right)$$

- This equation defines 6 independent components of a symmetric, second order tensor ($x_i \in \{x,y,z\}$, $u_i \in \{u,v,w\}$)

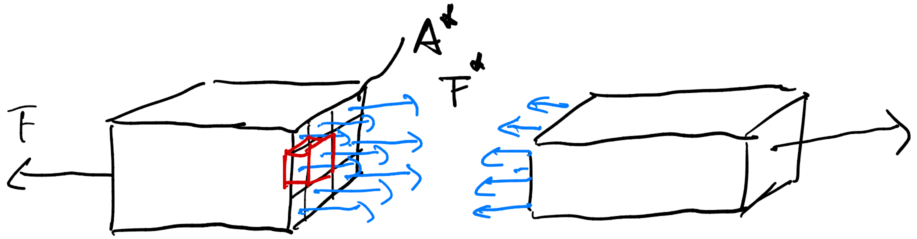


The Stress Tensor



MACROSCOPIC STRAIN :

$$\sigma := \frac{F}{A}$$

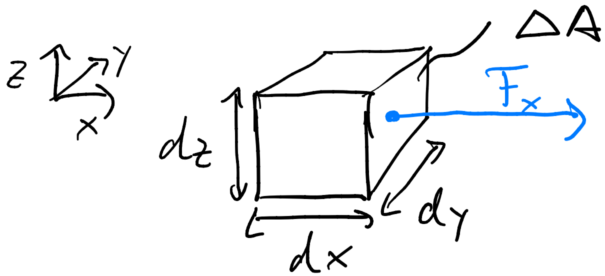


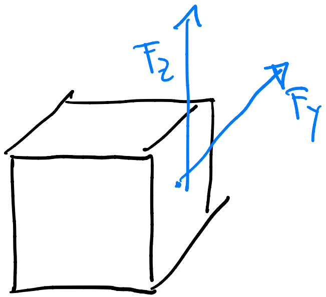
$$F = \sum F^* = 9F^*$$

$$A = \sum A^* = 9A^*$$

MACROSCOPIC STRESS:

$$\sigma = \frac{F}{A} = \frac{\cancel{9}F^*}{\cancel{9}A^*} = \frac{F^*}{A^*} \xrightarrow[\lim_{A^* \to 0}]{} \sigma = \lim_{\Delta A \to 0} \frac{F}{\Delta A}$$



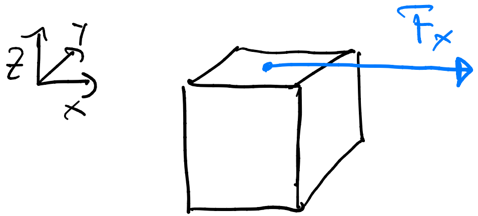


$\vec{F}$ IS A VECTOR

$$\sigma_{\sqcup x} = \frac{F_x}{\Delta A} = \frac{F_x}{d_y d_z} \qquad \text{EQN } 1$$

$$\left.\begin{aligned} \sigma_{\sqcup y} &= \frac{F_y}{\Delta A} = \frac{F_y}{d_y d_z} \\ \sigma_{\sqcup z} &= \frac{F_z}{\Delta A} = \frac{F_z}{d_y d_z} \end{aligned}\right\} \quad \text{EQN } 2$$

$\sigma_{\sqcup y}, \sigma_{\sqcup z} \Rightarrow$ SHEAR STRESSES



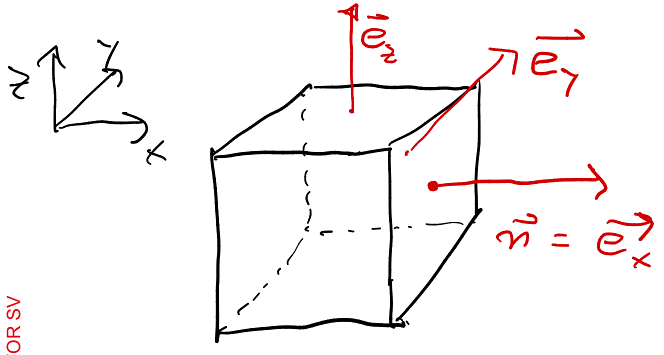
- FORCE COMPONENT IN THE x-DIRECTION WILL CREATE A STRESS

$$\sigma_{\perp x} = \frac{F_x}{\Delta A} = \frac{F_x}{dx \cdot dy} \qquad \text{EQN } (9)$$

↳ SHEAR STRESS

CONCLUSION: NOT ONLY THE DIRECTION THAT A FORCE IS ACTING MAKES A DIFFERENCE FOR THE STRESS, BUT ALSO WHICH FACE IT ACTS ON!

- WE THEREFORE DEFINE THE FACES OF THE VOLUME $dV = dx \cdot dy \cdot dz$ BY THE NORMAL VECTORS OF THE SURFACES



- THE FACE WE LOOKED AT IN THE FIRST EQN CAN THEN BE DESCRIBED AS:

$$\Delta A = \Delta A_x \cdot \vec{e}_x$$

- SINCE THE FORCE $\vec{F}$ IS ALSO A VECTOR, THE x-COMPONENT F_x CAN BE DESCRIBED AS:

$$F_x = \vec{F} \cdot \vec{e}_x$$

- EQN 1 THEN BECOMES

$$\sigma_{\perp x} = \frac{\vec{F} \cdot \vec{e}_x}{\| \Delta A_x \vec{e}_x \|} = \frac{\vec{F} \cdot \vec{e}_x}{\Delta A_x}$$

$$\sigma_{xx} = \frac{\vec{F}\, \vec{e}_x}{\Delta A_x}$$

- EQN 2:

$$\sigma_{xy} = \frac{\vec{F}\, \vec{e}_y}{\Delta A_x} \qquad \sigma_{xz} = \frac{\vec{F}\, \vec{e}_z}{\Delta A_x}$$

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- EQN 3 HOWEVER IS THEN:

$$\sigma_{zx} = \frac{\vec{F}\vec{e}_x}{\Delta A_z}$$

- BECAUSE WE LOOK AT AN INFINITESIMALLY SMALL ΔA:

$$\boxed{\sigma_{xx} = \lim_{\Delta A_x \to 0} \frac{\vec{F}\vec{e}_x}{\Delta A_x}}$$

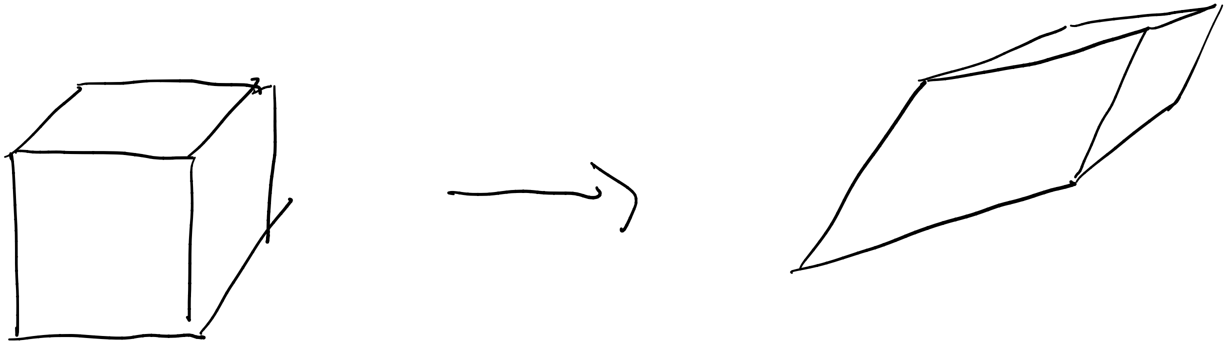
NAMING CONVENTION:

- FIRST SUBSCRIPT DEFINES THE NORMAL TO THE PLANE WHERE THE STRESS ACTS
- SECOND SUBSCRIPT DEFINES THE DIRECTION IN WHICH THE FORCE ACTS:

$$\sigma_{ij} = \lim_{\Delta A_i} \frac{F_j}{\Delta A_i} = \lim_{\Delta A_i \to 0} \frac{\vec{F} \cdot \vec{e}_j}{\Delta A_i}$$

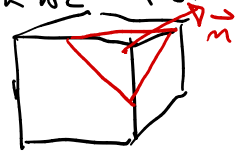
PLANE NORMAL

DIRECTION OF FORCE



$$\vec{\sigma}_x = \begin{pmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{xz} \end{pmatrix} \qquad \vec{\sigma}_y = \begin{pmatrix} \sigma_{yx} \\ \sigma_{yy} \\ \sigma_{yz} \end{pmatrix} \qquad \vec{\sigma}_z = \begin{pmatrix} \sigma_{zx} \\ \sigma_{zy} \\ \sigma_{zz} \end{pmatrix}$$

IF WE WANT TO KNOW THE STRESSES ON A PLANE THAT is NOT PART OF THE X, Y, OR Z PLANE WE USE THE GENERAL FORM:



$$\boxed{\vec{\sigma}_n = \lim_{\Delta A_n \to 0} \frac{\vec{F}}{\Delta A_n}}$$

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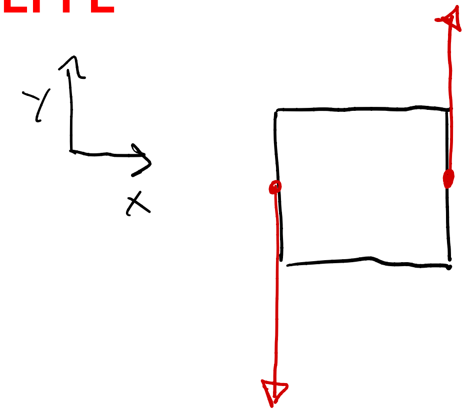
IN KARTHESIAN COORDINATES WE HAVE:

$$\sigma_{xx},\ \sigma_{xy},\ \sigma_{xz},\ \sigma_{yx},\ \sigma_{yy},\ \sigma_{yz},\ \sigma_{xz},\ \sigma_{yz},\ \sigma_{zz}$$

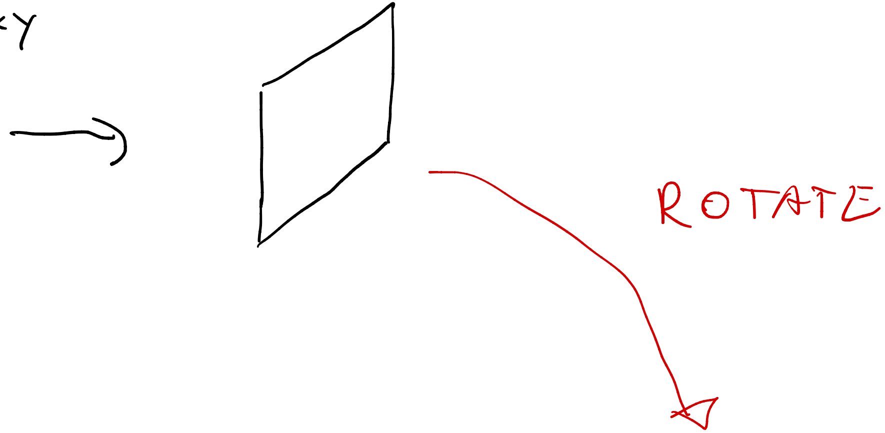
$\Rightarrow$ 9 STRESSES IN ONE INFINITESIMALLY SMALL ELEMENT

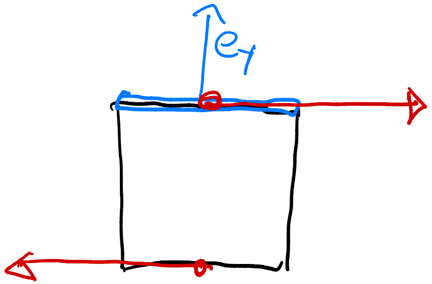
$\Rightarrow$ SUM UP INTO STRESS TENSOR.

$$\overleftrightarrow{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

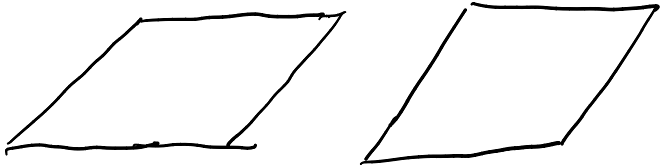


σ_{xy}





σ_{yx}



$$\sigma_{xy} = \sigma_{yx}$$

$\Rightarrow$ $\boxed{\text{STRESS TENSOR is SYMMETRIC}}$

The stress tensor

Nomenclature

- We can write the individual components of the stress state also in tensor form
- The first sub script denotes the normal vector of the area in question, and the second subscript denotes the direction of the force component that acts on the area

$$\sigma_{yx} = \lim_{\Delta A_y \to 0} \frac{\Delta F_x}{\Delta A_y} \qquad \sigma_{yy} = \lim_{\Delta A_y \to 0} \frac{\Delta F_y}{\Delta A_y}$$

- The stress vector equation can then be written as:

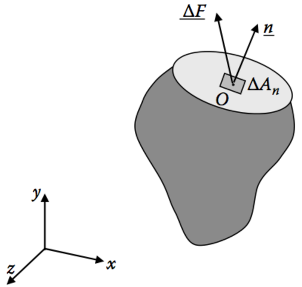
$$\vec{\sigma}_x = \sigma_{xx} \cdot \vec{e}_x + \sigma_{xy} \cdot \vec{e}_y + \sigma_{xz} \cdot \vec{e}_z$$
$$\vec{\sigma}_y = \sigma_{yx} \cdot \vec{e}_x + \sigma_{yy} \cdot \vec{e}_y + \sigma_{yz} \cdot \vec{e}_z$$
$$\vec{\sigma}_z = \sigma_{zx} \cdot \vec{e}_x + \sigma_{zy} \cdot \vec{e}_y + \sigma_{zz} \cdot \vec{e}_z$$

The stress tensor

Intensity of forces on internal surfaces

- *Stress is the intensity of internal forces (**F/A**) on (virtual) surfaces within a body subject to loads **P**. (**F**, **A**, and **P** are vectors)*
- *We can therefore define the stress vector $\underline{\sigma}_n$ as:*

$$\vec{\sigma_n} = \lim_{\Delta A_n \to 0} \frac{\Delta \vec{F}}{\Delta A_n}$$



- *In Cartesian coordinates:*

$$\vec{\sigma_n} = \lim_{\Delta A_n \to 0} \frac{\Delta F_x \vec{e_x} + \Delta F_y \vec{e_y} + \Delta F_z \vec{e_z}}{\Delta A_n}$$

- *For surfaces perpendicular to the x,y or z axis:*

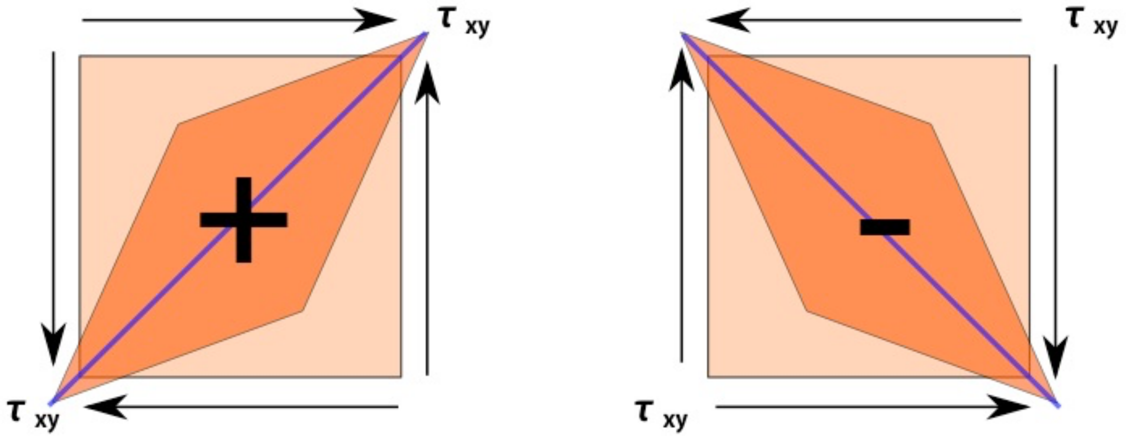
$$\vec{\sigma_x} = \lim_{\Delta A_x \to 0} \frac{\Delta F_x \vec{e_x} + \Delta F_y \vec{e_y} + \Delta F_z \vec{e_z}}{\Delta A_x}$$

$$\vec{\sigma_y} = \lim_{\Delta A_y \to 0} \frac{\Delta F_x \vec{e_x} + \Delta F_y \vec{e_y} + \Delta F_z \vec{e_z}}{\Delta A_y}$$

$$\vec{\sigma_z} = \lim_{\Delta A_z \to 0} \frac{\Delta F_x \vec{e_x} + \Delta F_y \vec{e_y} + \Delta F_z \vec{e_z}}{\Delta A_z}$$

The stress tensor – sign convention

- Normal stress: normal stress is considered positive if it puts an element in tension, and negative if it puts an element in compression
- Shear stress:



The stress tensor

Nomenclature

- The 9 stress components can be again combined to a tensor:

$$\overleftrightarrow{\tau} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} \sigma_{x} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{y} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{z} \end{pmatrix}$$

- σ_i are normal stresses, τ_{ij} are shear stresses
- The stress tensor is symmetric: $\tau_{xy} = \tau_{yx}$
- Both stress and strain tensor are 2nd order tensors and can be diagonalized to give the *principal values* (extreme values of stress and strain)

$$\begin{pmatrix} \sigma'_{xx} & 0 & 0 \\ 0 & \sigma'_{yy} & 0 \\ 0 & 0 & \sigma'_{zz} \end{pmatrix} \longrightarrow \text{PRINCIPAL COORDINATE SYSTEM}$$

The stress tensor

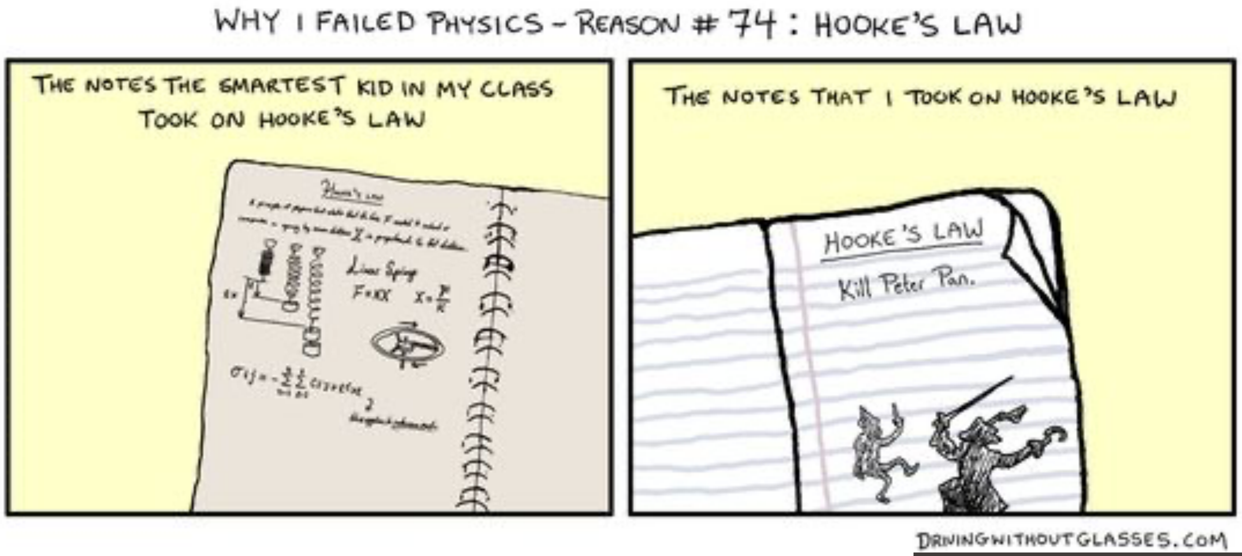
Final remarks

If we pass three mutually orthogonal planes through any given point O and find the stress vectors on each of the three mutually perpendicular faces drawn through O, then we have fully characterized the stress at point O

Due to the symmetry of the stress tensor, we only need 6 stress values to fully describe the stress state

Stress represents the intensity of internal forces on surfaces within a body subject to loads. At an imaginary cut or section, a vector sum of these forces keeps the body in equilibrium

Hooke's Law in 3D



Hooke's Law in 3D

- In the 1D case, Hooke's law has the form:

$$\sigma = E \cdot \varepsilon$$

$$\tau = G \cdot \gamma$$

- In 3D we do the same thing, but with ε,γ,σ,τ replaced by the second order tensors **ε** and **τ**:

$$\overleftrightarrow{\tau} = \overset{\Leftrightarrow}{C} \overleftrightarrow{\varepsilon}$$

$$\begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix} = \overset{\Leftrightarrow}{C} \begin{pmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{pmatrix}$$

- Or in index notation: $\tau_{ij} = C_{ijkl} \cdot \varepsilon_{kl}$
- C_{ijkl} is called the *stiffness*, C_{ijkl}^{-1} is called the *compliance*
- A tensor of rank 4 in 3D space has 3x3x3x3=**81 elements**!

Georg Fantner

$$\overleftrightarrow{\varepsilon} = \overleftrightarrow{C}^{-1}\,\overleftrightarrow{\tau}$$

$$\begin{pmatrix} \varepsilon_{xx} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_{yy} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_{zz} \end{pmatrix} = \overleftrightarrow{C}^{-1} \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{pmatrix}$$

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{pmatrix} = \begin{pmatrix} 6 \times 6 \\ \text{MATRIX} \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{pmatrix}$$

- WHAT STRESSES CAN INFLUENCE THE STRAIN IN THE x-DIRECTION (ε_{xx})?

- NORMAL STRESSES IN x-DIRECTION: $\varepsilon_{xx} = \frac{1}{E}\sigma_{xx}$
- THROUGH POISSON EFFECT: $\boxed{\varepsilon_{xx} = -\nu\,\varepsilon_{yy}}$
  - NORMAL STRESS IN y $\quad \varepsilon_{xx} = -\nu\left(\frac{1}{E}\sigma_{yy}\right)$
  - NORMAL STRESS IN z $\quad \varepsilon_{xx} = -\nu\left(\frac{1}{E}\sigma_{zz}\right)$

$$+$$

$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy} - \nu\sigma_{zz}\right)$$

- SIMILARLY WE GET:

$$\varepsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu\sigma_{xx} - \nu\sigma_{zz}\right)$$

$$\varepsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - \nu\sigma_{xx} - \nu\sigma_{yy}\right)$$

– SHEAR STRESSES :

$$\gamma_{xy} = \frac{1}{G}\,\sigma_{xy} \qquad G = \frac{E}{2(1+\nu)}$$

$$\gamma_{xy} = \frac{1}{E}\,2(1+\nu)\,\sigma_{xy}$$

$$\gamma_{xz} = \frac{1}{E}\,2(1+\nu)\,\sigma_{xy}$$

$$\gamma_{yz} = \frac{1}{E}\,2(1+\nu)\,\sigma_{yz}$$

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{pmatrix} = \frac{1}{E} \begin{pmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{pmatrix}$$

Generalized Hooke's law

Simplification of the stiffness matrix

Mathematically general form	81 independent components	
Take into account the symmetry of τand ε	36 independent components	
symmetry due to the existence of strain energy U_0	21 independent components	General case for anisotropc elasticity
isotropic case: C_{ijkl} is invariant to coordinate rotation	2 independent components	Case for isotropic materials (materials that have same E and G in all directions)

Generalized Hooke's law

Full form for isotropic homogeneous materials

- Because of the symmetry we can simplify:

$$\begin{pmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{pmatrix} = \overset{\Leftrightarrow}{C}^{-1} \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix}$$

- We rearrange the non redundant components:

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{pmatrix} = \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ 2\varepsilon_{xy} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{yz} \end{pmatrix} = \begin{pmatrix} \text{some 6x6 marix} \\ \\ \\ \end{pmatrix} \cdot \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau yz \end{pmatrix}$$

- These are no longer the stress or strain tensors! They are just a simpler way to write Hooke's law in 3D.

EPFL

Generalized Hooke's law

For isotropic cases

- For homogeneous isotropic materials we can write Hooke's law for the normal stresses and for the shear stresses as:

$$\varepsilon_x = \varepsilon_{xx} = \frac{1}{E}(\sigma_x - \nu \cdot \sigma_y - \nu \cdot \sigma_z)$$
$$\varepsilon_y = \varepsilon_{yy} = \frac{1}{E}(-\nu \cdot \sigma_x + \sigma_y - \nu \cdot \sigma_z)$$
$$\varepsilon_z = \varepsilon_{zz} = \frac{1}{E}(-\nu \cdot \sigma_x - \nu \cdot \sigma_y + \sigma_z)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$
$$\gamma_{xz} = \frac{\tau_{xz}}{G}$$
$$\gamma_{yz} = \frac{\tau_{yz}}{G}$$

- Since E and G are related through v, there are only 2 independent materials properties in these equations.

$$G = \frac{E}{2(1+\nu)}$$

Generalized Hooke's law

Full form for isotropic homogeneous materials

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{pmatrix} = \frac{1}{E} \begin{pmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{pmatrix} \cdot \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{pmatrix}$$

Generalized Hooke's law

Volume changes

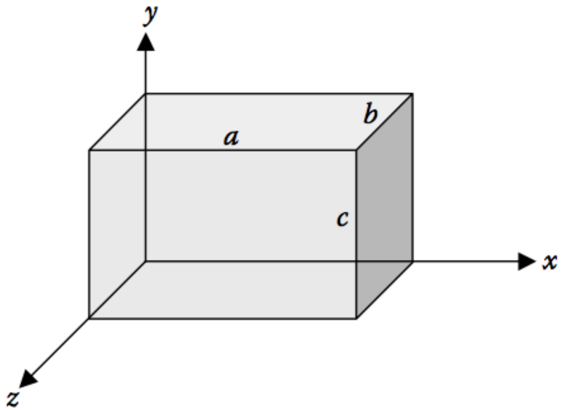
- From the definition of the infinitesimal strains, we can calculate the change in volume to be

$$\begin{aligned}\frac{\Delta V}{V} &= \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \\ &= \frac{1-2\nu}{E}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) \\ &= \frac{1}{K}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})\end{aligned}$$

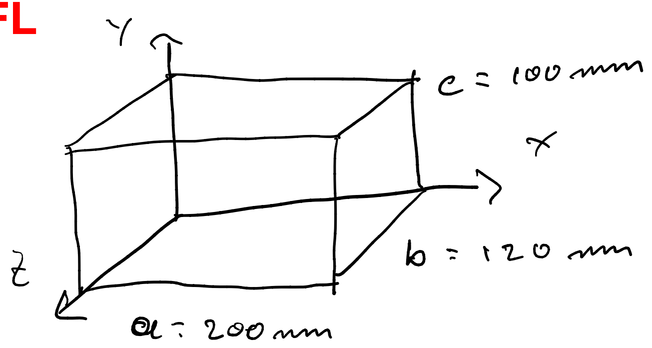
- *K* is the *bulk modulus* of the material. It can be calculated from the two independent materials variables E and v

Example 3.1

Triaxial loading



A rectangular copper alloy block as shown in the figure has the following dimensions: *a* = 200 mm, *b* = 120 mm, and *c* = 100 mm. This block is subjected to a triaxial loading in equilibrium having the following magnitude: σx = +2.40 MPa, σy = –1.20 MPa, and σz = –2.0 MPa. Assuming that the applied forces are uniformly distributed on the respective faces, determine the size changes that take place along *a, b,* and *c.* Let *E* = 140 GPa and v = 0.35.



GIVEN: ▫ STRESS STATES: $\sigma_{xx} = 2.4\ MPa$

$\sigma_{yy} = -1.2\ MPa$

$\sigma_{zz} = -2\ MPa$

▫ MAT. PROP.

$E = 140\ GPa$

$\nu = 0.35$

▫ DIMENSIONS

$a = 200\ mm$

$b = 120\ mm$

$c = 100\ mm$

ASKED: $\delta_a, \delta_b, \delta_c$

GOV. PRINC.: HOOKE'S LAW IN 3D

ANSWER:

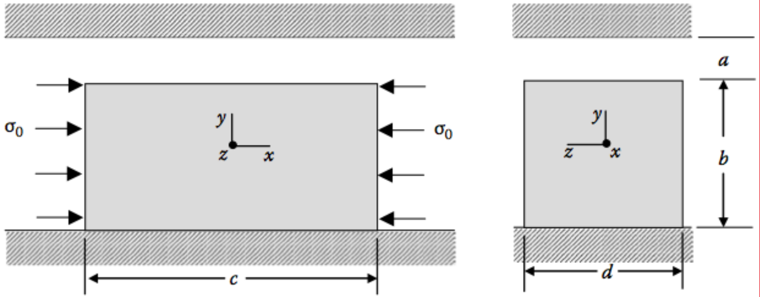
$$\varepsilon_{xx} = \frac{\delta_a}{a} = \frac{1}{E}\left(\sigma_{xx} - \nu\,\sigma_{yy} - \nu\,\sigma_{zz}\right)$$

$$= \frac{1}{140 \cdot 10^{9}}\left(2.4 + 0.35 \cdot 1.2 + 0.35 \cdot 2\right) \cdot 10^{6} = 25.1 \cdot 10^{-6}$$

$$\delta_a = \varepsilon_{xx} \cdot a = 25.1 \cdot 10^{-6} \cdot 200\ \text{mm} = 5.03 \cdot 10^{-3}\ \text{m} = 5{,}03\ \mu\text{m}$$

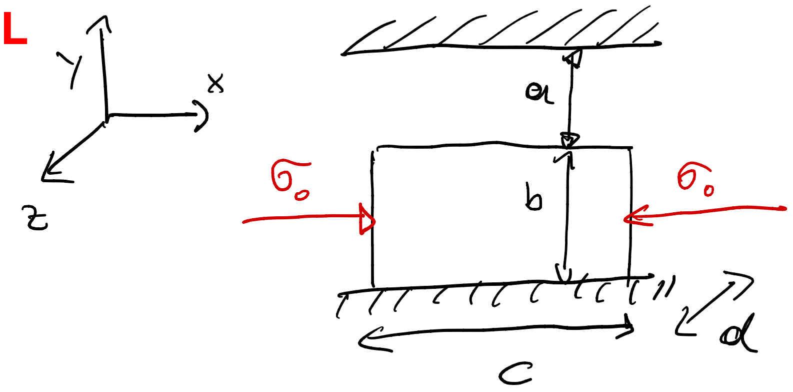
$$\delta_b = -950\ \text{nm}$$

$$\delta_c = -2{,}10\ \mu\text{m}$$



Example 3.2

A rectangular block is compressed by a uniform stress σ_0 as it sits between two rigid surfaces with the gap *a* shown in Figure 3.14. Determine (a) the stress σ_{yy}; (b) the change in the length along the *x* axis as the gap *a* is closed; and (c) the minimum value of σ_0 need to close the gap and the change in length of c when the gap is just closed. Assume $\sigma_{zz}=0$



given: a geometry · b, c, d

· gap a

□ Load: $\sigma_{xx} = \sigma_0$ so that gap closes $\delta_b = a$

□ assume: $\sigma_{zz} = 0$

ASKED:

a) σ_{yy}

b) δ_c

c) σ_0 $\quad \delta_c \big|_{gap=0}$

EPFL

GOV. PRINC: 3D HOOKE'S LAW

Georg Fantner

ANSWER:

a) $\varepsilon_{yy} = \frac{\delta_b}{b}$ WHEN GAP IS CLOSED $\varepsilon_{yy} = \frac{a}{b}$

THEREFORE $\forall$ $\boxed{\delta_b < a : \quad \sigma_{yy} = 0 \,!}$

$\delta_b = a$: $\varepsilon_{yy} = \frac{1}{E}\left(-\nu\sigma_{xx} + \sigma_{yy} - \nu\cancel{\sigma_{zz}}\right)$ $= 0$

$\sigma_{yy} = E\varepsilon_{yy} + \nu\sigma_{xx}$

$= E\varepsilon_{yy} - \nu\sigma_0$

$$\boxed{\sigma_{yy} = E\frac{a}{b} - \nu\sigma_0}$$

$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy} - \nu\cancel{\sigma_{zz}}\right)$$

$$= \frac{1}{E}\left(-\sigma_0 - \nu\left(E\frac{a}{b} - \nu\sigma_0\right)\right)$$

$$= \frac{1}{E}\left((\nu^2-1)\sigma_0 - E\nu\frac{a}{b}\right)$$

$$\boxed{\delta_c = \varepsilon_{xx}\cdot c = \frac{c}{E}\Bigl(\nu^2-1\Bigr)\sigma_0 - \nu\frac{ac}{b}}$$

PART C:

FOR JUST CLOSING THE GAP $\sigma_{YY} = 0$

$$E \frac{a}{b} - \nu \sigma_0 = 0 \Rightarrow \sigma_0 = \frac{E a}{b \nu}$$

$\delta_c \big|_{a=0}$:

$$\delta_c \big|_{a=0} = \frac{c}{\cancel{E}} (\nu^2 - 1) \frac{\cancel{E} a}{b \nu} - \nu \frac{a c}{b}$$

$$= \cancel{\frac{a c \nu}{b}} - \frac{a c}{b \nu} - \cancel{\frac{a c \nu}{b}}$$

$$\boxed{\delta_c \big|_{a=0} = - \frac{a c}{b \nu}}$$